3 Differentiation

The differential calculus has two major areas of use and origin. One is geometry, and the problem of finding tangents to curves. The other is motion (speed, velocity, acceleration) and other rates of change. Both of these lead to the definition of the derivative in terms of a limit.

3.1 The Limit Definition

We shall explore the definition of derivative by considering the problem of finding the gradient of a curve, and therefore its tangent.

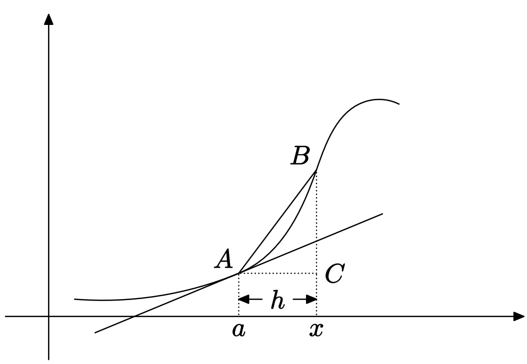

Figure 3.1 Chord slope diagram

Figure 3.1 represents the graph of a function $y = f(x)$. The tangent line at the point A can be considered as the limiting position of the chord AB as B tends towards A. This is achieved by letting x tend to a, or equivalently by letting h tend to 0, since $x = a + h$. It is important to note that this is not just a one-sided limit, and so a diagram where B is to the left of A, where h is negative, is equally valid. The gradient of the tangent will therefore be the limit as x tends to a of the gradient of the chord. The coordinates of the points labelled in Figure 3.1 are as follows.

$$A(a, f(a)),\ B(x, f(x)),\ C(x, f(a)).$$

Therefore the gradient (slope) of the chord is given by

$$\frac{BC}{AC} = \frac{f(x) - f(a)}{x - a} = \frac{f(a+h) - f(a)}{h}.$$

Taking limits therefore gives the gradient of the tangent, giving rise to the following definition.

Definition 3.1

The function $f(x)$ whose domain includes some interval containing the point a is said to be **differentiable** at a if the following limit exists.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

The value of this limit is called the **derivative** (or **differential coefficient**) of f at a, denoted by $f'(a)$. We therefore have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

In many cases a function will be differentiable for all (or most) of the values of x in the domain. In that case we think of a as a variable and use the term **derivative** for the function whose value at $x = a$ is $f'(a)$.

In fact a variety of terminology is encountered in this topic. Terms such as differential coefficient, derived function, differential and derivative are all used, sometimes to convey different shades of meaning and interpretation. In an introductory account such distinctions are not so important, whereas they are in more advanced areas of calculus. We shall use the term derivative to refer to the function resulting from the process of differentiation (sometimes called the **derived function**), and also to the value of this derived function at some point of its domain.

There are two types of notation in common use, the dash notation $f'(x)$, $f''(x)$, etc., and the Leibniz notation $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$, etc. Readers will have encountered these in school calculus. Sometimes the Leibniz notation is more helpful than the dash notation, and vice-versa, and we shall use them according to this criterion.

3.2 Using the Limit Definition

In this section we consider some examples where we can find the derivative directly from the limit, together with an example where the derivative does not exist. We also prove a basic property of derivatives useful in graph sketching. In practice we do not rely heavily on the limit definition. Instead we use algebraic rules for differentiation and apply them to functions whose derivatives we already know. This mirrors the procedure we used in Chapter 2 to find limits, and the processes of integration we shall develop in later chapters. The first two examples show that the basic derivatives can be found using the limit definition. This is sometimes referred to as "finding the derivative from first principles", the first principles in question being the limit definition.

Example 3.2

Use the limit definition to find the derivative of $f(x) = x^2$.

Applying the limit definition gives

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} (x + a) = 2a.$$

Example 3.3

Use the limit definition to find the derivative of $f(x) = \sin x$.

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\sin(a+h) - \sin a}{h} \\ &= \lim_{h \to 0} \frac{2\cos\left(a + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = \lim_{h \to 0} \cos\left(a + \frac{h}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \\ &= \cos(a).1 = \cos a. \end{aligned}$$

Here we have used the limit obtained in Example 2.2, with $x = \frac{h}{2}$.

Example 3.4

Show that $f(x) = |x|$ is not differentiable at 0.

We recall the graph of $y = |x|$ shown in Figure 1.8, and notice that it has a sharp corner at $x = 0$. The gradient to the right is 1 and the gradient to the left is -1, indicating that the gradient at 0 cannot be well-defined. The limit definition confirms this, as follows.

$$\frac{f(0+h) - f(0)}{h} = \begin{cases} \frac{h}{h} = 1 & \text{for } h > 0, \\ \frac{-h}{h} = -1 & \text{for } h < 0. \end{cases}$$

So the left- and right-sided limits are different. Therefore the (two-sided) limit does not exist, and so the modulus function is not differentiable at 0.

Example 3.5

In Section 1.8 we considered functions defined in pieces. In that section we introduced the function

$$k(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ x^3 & \text{if } x < 0. \end{cases}$$

If $x > 0$ then $k(x) = x^2$ and so $k'(x) = 2x$. If $x < 0$, $k(x) = x^3$ and so $k'(x) = 3x^2$. But to investigate differentiability at $x = 0$ we need to use the limit definition, as follows.

$$\frac{k(0+h) - k(0)}{h} = \begin{cases} \frac{h^2-0}{h} = h & \text{if } x > 0, \\ \frac{h^3-0}{h} = h^2 & \text{if } x < 0. \end{cases}$$

We conclude from this that

$$\lim_{h \to 0^+} \frac{k(0+h) - k(0)}{h} = \lim_{h \to 0^+} h = 0;$$

$$\lim_{h \to 0^-} \frac{k(0+h) - k(0)}{h} = \lim_{h \to 0^-} h^2 = 0.$$

The left- and right-hand limits are equal, so

$$\lim_{h \to 0} \frac{k(0+h) - k(0)}{h} = 0.$$

Therefore $k(x)$ is differentiable at $x = 0$ and $k'(x) = 0$.

Figure 3.2 shows the graph of $k'(x)$. We can see that there appears to be a sharp corner at $x = 0$, as there is for $|x|$. This suggests that $k'(x)$ is not

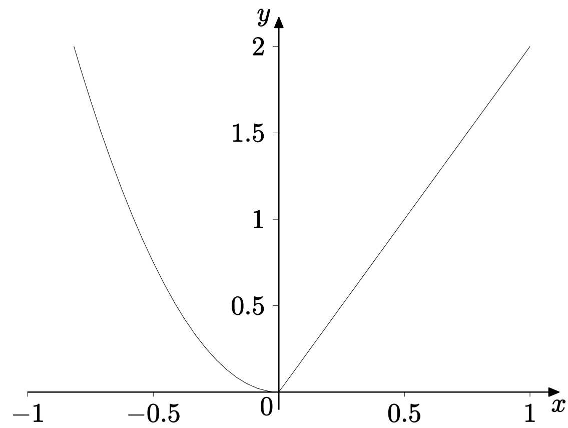

Figure 3.2 A piecewise derivative

differentiable at $x = 0$, and we can prove this using the limit definition. We have

$$\frac{k'(0+h) - k'(0)}{h} = \begin{cases} \frac{2h-0}{h} = 2 & \text{if } x > 0, \\ \frac{3h^2-0}{h} = 3h & \text{if } x < 0. \end{cases}$$

So we deduce that

$$\lim_{h \to 0^+} \frac{k'(0+h) - k'(0)}{h} = \lim_{h \to 0^+} 2 = 2;$$

$$\lim_{h \to 0^-} \frac{k'(0+h) - k'(0)}{h} = \lim_{h \to 0^-} 3h = 0.$$

The left-and right-hand limits are not the same, and so

$$\lim_{h \to 0} \frac{k'(0+h) - k'(0)}{h} \quad \text{does not exist.}$$

This shows that $k'(x)$ is not differentiable at $x = 0$.

Example 3.6

Prove that if a differentiable function is increasing (see Definition 1.35) then its derivative is non-negative.

Suppose that for all a, b in the domain of f satisfying $a \leq b$, we have $f(a) \leq f(b)$. Let x denote an arbitrary number in the domain of f. Then if $h > 0$ we have $\dfrac{f(x+h) - f(x)}{h} \geq 0$, because both numerator and denominator are positive (or zero). If $h < 0$ we also have $\dfrac{f(x+h) - f(x)}{h} \geq 0$, because in this case both numerator and denominator are negative (or zero). Therefore we must have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \geq 0.$$

We can prove in a very similar fashion that if a differentiable function is decreasing then its derivative is non-positive. Note that if a function is strictly decreasing this does not imply that its derivative is strictly positive everywhere. For example $f(x) = x^3$ is strictly increasing, but $f'(0) = 0$.

We shall prove a converse of this result under appropriate conditions in Section 6.2.

3.3 Basic Rules of Differentiation

The basic algebraic rules of differentiation enable us to differentiate sums, products and quotients of functions whose derivatives we already know. We assume that the derivatives in the table below are known from school mathematics.

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
e^x	e^x
$\ln x$	$\frac{1}{x}$

The basic rules of differentiation are summarised as follows.

Suppose that f and g are differentiable functions. Then for any constants A, B,

$$\frac{d}{dx}(Af(x) + Bg(x)) = Af'(x) + Bg'(x) \quad \text{(the sum rule);}$$

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x) \quad \text{(the product rule);}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \quad (g(x) \neq 0) \quad \text{(the quotient rule).}$$

This is a case where it is convenient to use both notations for derivatives together.

Example 3.7

We prove the product rule from the limit definition of the derivative.

In the second line of the proof below, we have introduced an additional term in the numerator, together with its negative, hence preserving equality. Its purpose is to enable us to analyse separately the changes in f and the changes in g as h tends to zero. This is apparent in the third line, where we can see the chord-slope quotients for f and for g, whose limits are the respective derivatives as h tends to zero.

$$\begin{aligned}
&\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \\
&= f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h} \\
&\to f(x)g'(x)+g(x)f'(x) \quad \text{as} \quad h \to 0.
\end{aligned}$$

3.4 The Chain Rule

The Chain Rule (or function of a function rule) tells us how to differentiate composite functions, and while it is usually part of school calculus, it is sufficiently important to merit some revision. The rule is stated as follows.

Suppose that the function g is differentiable at x, and that the function f is differentiable at $g(x)$. Then the derivative of $f \circ g(x) = f(g(x))$ is given by $(f \circ g)'(x) = f'(g(x))g'(x)$.

The rule can be stated using the Leibniz notation as follows.

If $y = f(u)$ and $u = g(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

To derive the chain rule from the limit definition we proceed as follows.

$$\frac{f(g(x+h))-f(g(x))}{h} = \frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\,\frac{g(x+h)-g(x)}{h}.$$

Here we have introduced the term $g(x+h)-g(x)$ in the numerator and in the denominator. This helps to separate the behaviour of f and that of g. We then let $g(x) = u$ and $g(x+h) = u+k$. Then $k \to 0$ as $h \to 0$. We therefore have

$$\begin{aligned}
&\frac{f(g(x+h))-f(g(x))}{h} = \frac{f(u+k)-f(u)}{k}\,\frac{g(x+h)-g(x)}{h} \\
&\to f'(u)g'(x) = f'(g(x))g'(x) \quad \text{as} \quad h \to 0.
\end{aligned}$$

This argument appears to be sound, but in fact there is a problem at the beginning, for we cannot be sure that $g(x+h)-g(x)$ will not be zero for some values of h arbitrarily close to 0, and we cannot divide by zero. The argument does however provide an intuitive justification relating the chain rule to the limit definition. A properly rigorous proof is given in Howie Chapter 4.

The use of an "intermediate variable" such as u in applying the chain rule is often helpful and we shall employ it in the following examples.

Example 3.8

Differentiate $\ln(\cos x)$.

Using the Chain Rule, let $y = \ln u,\ u = \cos x$. The derivative is given by

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}(-\sin x) = \frac{-\sin x}{\cos x} = -\tan x.$$

Example 3.9

Differentiate a^x with respect to x.

A common mistake is to assume that we use the simple formula for powers and write the derivative as xa^{x-1}. This is WRONG. What has been calculated here is the derivative with respect to a, not the derivative with respect to x. To do the calculation correctly, we recall from Definition 1.27 that $a^x = \mathrm{e}^{x \ln a}$.

We therefore use the chain rule, letting $y = \mathrm{e}^u$ and $u = x \ln a$. This gives

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \mathrm{e}^u \ln a = \mathrm{e}^{x \ln a} \ln a = a^x \ln a.$$

Example 3.10

Differentiate $\sin(\ln(x^3 - 4x))$.

In this example we have repeated composition, and we extend the chain rule using two intermediate variables.

We let $y = \sin u,\ u = \ln t,\ t = x^3 - 4x$. The derivative is then given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dt}\frac{dt}{dx} = (\cos u).\frac{1}{t}.(3x^2 - 4) \\ &= \frac{\cos\big(\ln(x^3 - 4x)\big)\,(3x^2 - 4)}{x^3 - 4x}.\end{aligned}$$

Example 3.11

Differentiate $\ln\left(\tan\left(2 + x^4\right)^{\frac{1}{2}}\right)$.

There is no limit to the number of stages of composition.

In this case we introduce the requisite number of intermediate variables, letting $y = \ln p,\ p = \tan q,\ q = r^{\frac{1}{2}},\ r = 2 + x^4$.

Applying the chain rule therefore gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dp}\frac{dp}{dq}\frac{dq}{dr}\frac{dr}{dx} = \frac{1}{p}.\sec^2 q.\frac{1}{2}r^{-\frac{1}{2}}.4x^3 \\ &= \frac{\sec^2\left(\left(2+x^4\right)^{\frac{1}{2}}\right).2x^3}{\tan\left(\left(2+x^4\right)^{\frac{1}{2}}\right)\left(2+x^4\right)^{\frac{1}{2}}}. \end{aligned}$$

Example 3.12

Differentiate $f(x) = x^2 \cos\left(\frac{1}{x}\right)$.

In examples such as this one we have to use more than one of the rules. Firstly we need the product rule since the function is x^2 multiplied by a cosine term. Secondly the cosine terms itself is composite, and so we need the chain rule. So applying both rules gives

$$f'(x) = 2x\cos\left(\frac{1}{x}\right) + x^2\left(-\sin\left(\frac{1}{x}\right)\right)\left(-\frac{1}{x^2}\right) = 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right).$$

Figure 3.3 shows a MAPLE plot of this formula for $f'(x)$. (We pointed out the limitations of MAPLE plots of such functions in Example 2.3)

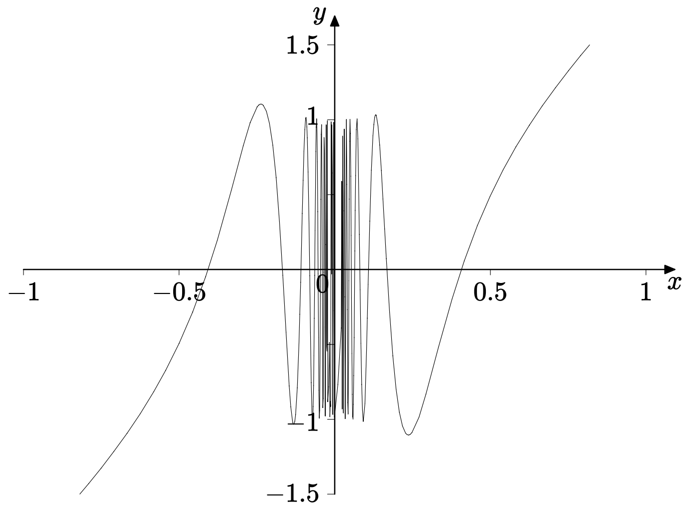

Figure 3.3 A discontinuous derivative

Now this calculation is not valid when $x = 0$, and indeed $f(x)$ is not defined there. However, the squeezing argument used in Example 2.13 shows that $f(x)$ has the limit zero as x tends to zero, and that if we extend the definition by letting $f(0) = 0$ the resulting function is continuous at 0. So what about differentiability? We can't substitute $x = 0$ in the formula we have just found, so we have to go back to the limit definition, and investigate whether the appropriate limit exists.

$$\frac{f(0+h) - f(0)}{h} = \frac{h^2 \cos\left(\frac{1}{h}\right)}{h} = h \cos\left(\frac{1}{h}\right) \to 0 \quad \text{as} \quad h \to 0,$$

by the squeezing argument. Therefore f is differentiable at 0 and $f'(0) = 0$. This is a very interesting example, because although f is differentiable at 0, we can see that the formula for $f'(x)$ does not have a limit as $x \to 0$, because the sin term oscillates infinitely often in any interval containing $x = 0$, as we established in Example 2.3. So f is differentiable everywhere but the derivative is discontinuous at $x = 0$.

3.5 Higher Derivatives

If we have a function $y = f(x)$ specified by a given formula and we differentiate it we obtain the formula for $f'(x)$, which we can usually differentiate again, and in many cases we can repeat the process several times. This gives a sequence of derivatives, denoted by

$$f'(x), f''(x), f'''(x), f^{(4)}(x), \ldots, f^{(n)}(x), \ldots,$$

or, using the Leibniz notation for derivatives,

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}, \ldots.$$

Higher derivatives have applications, for example in mechanics where the second derivative of position relates to acceleration, and in coordinate geometry as we shall see in Chapter 5.

Example 3.13

Find the n-th derivative of $f(x) = \ln(2x + 3)$.

Calculating the first few derivatives, using the chain rule, is relatively straightforward, giving

$$f(x) \quad = \quad \ln(2x + 3);$$

$$\begin{aligned}
f'(x) &= \frac{2}{2x+3};\\
f''(x) &= \frac{-4}{(2x+3)^2};\\
f'''(x) &= 2.\frac{8}{(2x+3)^3};\\
f^{(4)}(x) &= 3.2.\frac{-16}{(2x+3)^4};\\
f^{(5)}(x) &= 4.3.2.\frac{32}{(2x+3)^5}.
\end{aligned}$$

There is a clear pattern appearing here, which enables us to conjecture a formula for the n-th derivative, namely

$$f^{(n)}(x) = (n-1)!\frac{(-1)^{(n+1)}2^n}{(2x+3)^n}.$$

To prove this we need to use the method of mathematical induction. Readers who are not familiar with this method can normally rely on conjecturing such results by generalising, without proof, the pattern observed in the first few cases. The details of the method are discussed in Howie Chapter 1. The inductive proof is as follows, for readers who are familiar this style of proof.

The case $n = 1$ has already been established. If the result were true for $n = k$ then

$$f^{(k)}(x) = (k-1)!\frac{(-1)^{(k+1)}2^k}{(2x+3)^k} = (k-1)!(-1)^{(k+1)}2^k(2x+3)^{-k}.$$

Differentiating this formula once more would give

$$f^{(k+1)}(x) = (k-1)!(-1)^{(k+1)}2^k(-k)(2x+2)^{-k-1}.2 = k!\frac{(-1)^{(k+2)}2^{k+1}}{(2x+3)^{k+1}},$$

giving the result for $n = k+1$, so proving the general result by induction.

Example 3.14

Find successive derivatives for $f(x) = \sin\left(x^2\right)$.

Using the chain rule and the product rule gives

$$\begin{aligned}
f(x) &= \sin\left(x^2\right);\\
f'(x) &= 2x\cos\left(x^2\right);\\
f''(x) &= -4x^2\sin\left(x^2\right) + 2\cos\left(x^2\right).
\end{aligned}$$

It can be seen even at this stage that the expressions are increasing in complexity, so that for example the sixth derivative is given by

$$f^{(6)}(x) = -64x^6 \sin\left(x^2\right) + 480x^4 \cos\left(x^2\right) + 720x^2 \sin\left(x^2\right) - 120 \cos\left(x^2\right).$$

While it is clear that there is a pattern here, it is not easy to formulate an expression for the n-th derivative.

3.6 Differentiation using MAPLE

The MAPLE command for differentiation is straightforward. For example executing the command `diff(sin(x^2),x);` will give the derivative of $\sin\left(x^2\right)$. The role of the `x` at the end of the command it important. An error message will result if it is omitted. It tells MAPLE what the variable of differentiation is. This can be illustrated with the two commands

`diff(x^2*y^3,x);` and `diff(x^2*y^3,y);`

which will return the outputs $2xy^3$ and $3x^2y^2$ respectively.

It should be noted that the output from MAPLE will not necessarily look identical to a formula we would obtain "by hand". For example the command `diff(tan(x),x);` produces $1 + \tan^2 x$ instead of $\sec^2 x$, but the two are of course identical. The command `simplify(%);` will sometimes transform an answer into a more recognisable form. The percentage (`%`) symbol denotes "the previous expression" or "the output of the previous calculation", and using it can save having to repeat typing in a complicated expression.

Calculating successive derivatives is also straightforward using MAPLE. One can enter a formula such as `sin(x^2);` Executing this command simply prints the formula $\sin\left(x^2\right)$ on the screen. The command `diff(%,x);` then calculates the derivative with respect to x. Following this with the same command `diff(%,x);` will therefore repeat the process, giving the second derivative. This procedure could be used to calculate the sixth derivative quoted in Example 3.14. Alternatively we could calculate the sixth derivative directly using the command

```
diff(sin(x^2),x$6);
```

where the `$6` sign tells MAPLE that we want the sixth derivative.

Finally we note that one need not have a particular formula, so that for example MAPLE will help us if we forget the product rule. Entering the command `diff(f(x)*g(x),x);` produces the output

$$\left(\frac{d}{dx} f(x)\right) g(x) + f(x) \left(\frac{d}{dx} g(x)\right).$$

EXERCISES

3.1. Use the limit definition to find the derivative of each of the following functions

(a) x^3; (b) x^{-1}; (c) $\cos x$; (d) $\tan x$; (e) e^x.

3.2. Use the sum, product and quotient rules to find the derivative of each of the functions defined by the following expressions.

(a) $8x^{3/4}$; (b) $\sinh x$; (c) $\mathrm{e}^v \sin v$;

(d) $x^2 \tan x$; (e) $t \sin t + \cos t$; (f) $\tanh x$;

(g) $\dfrac{3x-2}{2x-3}$; (h) $\dfrac{t^2+2t}{t^2-1}$; (i) $\dfrac{1-4x}{x^{2/3}}$;

(j) $\dfrac{\cos x}{1+2\sin x}$; (k) $\dfrac{\mathrm{e}^w}{1-\tan w}$; (l) $\dfrac{\mathrm{e}^x \ln x}{x^2+2x^3}$.

3.3. Use the chain rule to find the derivative of each of the functions defined by the following expressions.

(a) $\cos\left(\sqrt{x}\right)$; (b) $\cosh(\cos t)$; (c) 2^{-x};

(d) $\ln(\ln(\ln x))$; (e) $\left(1+s^{2/3}\right)^{3/2}$; (f) $\left(3-2t^2\right)^{-3/4}$;

(g) $\tan\left(\frac{1}{x}\right)$; (h) $\sqrt{\sin\left(v^2\right)}$; (i) $\sin(2\cos 3x)$;

(j) 3^{3^x}; (k) $\cos(\ln x)$; (l) $\sqrt[3]{\ln t}$.

3.4. Find the derivative, with respect to x, of each of the functions defined by the following expressions, using the appropriate combinations of rules.

(a) $\ln(x \sin x)$; (b) $\sin\left(\dfrac{x}{\cos x}\right)$;

(c) $\sqrt{x+\mathrm{e}^x}$; (d) $\sqrt{x}\sqrt[3]{1+x^2}$;

(e) $\cosh(x \ln x)$; (f) $\dfrac{\sin\left(x^2\right)}{\sec\left(x^2\right)}$;

(g) $\tan(3x^3)\cot(3x^3)$; (h) $\tan\left(a^2\left(1+x^2\right)\right)$;

(i) $2^{x \sin x}$; (j) $a\sin(bx) + b\sin(ax)$;

(k) $(x^2 \ln x)^{(b^2)}$; (l) $\tan^2\left(\dfrac{1}{cx^2+d}\right)$.

3.5. Find successive derivatives of $f(x) = \sin(2x - 5)$. On the basis of the first few derivatives write down a general formula for $f^{(2n)}(x)$ and for $f^{(2n+1)}(x)$. Prove these results by the method of mathematical induction.

Challenge: find a single formula which covers both the separate formulae above.

3.6. Use MAPLE to find successive derivatives of $f(x) = \mathrm{e}^{ax} \sin ax$. Write down general formulae for $f^{(4n)}(x)$, $f^{(4n+1)}(x)$, $f^{(4n+2)}(x)$ and $f^{(4n+3)}(x)$. Prove these results by the method of mathematical induction.

3.7. The derivative of an even function is an odd function.

The derivative of an odd function is an even function.

(a) Write a clear explanation of these results based on diagrams.

(b) Prove the results by differentiating the equations which define an even function and an odd function, given in Definitions 1.4 and 1.5.

(c) Prove the results from the limit definition of the derivative, given in Definition 3.1.

Do you think the converses are true, namely that every odd (even) function is the derivative of an even (odd) function? If you think so, give a proof. If you do not think so, give a counter-example. If you think the converses are true only for some kinds of function, describe such a set of functions and prove the converses for this set.