4
Techniques of Differentiation

In this chapter we shall explore some techniques of differentiation which deal with functions specified in various forms. We shall consider functions defined implicitly, functions defined parametrically, functions involving powers, and inverse functions. We shall also discuss Leibniz Theorem, a result which enables higher derivatives of products to be calculated.

4.1 Implicit Differentiation

Sometimes we are not given y as a function of x explicitly, but instead have an equation connecting them which we may be unable to solve explicitly for either x or y. We may still want to find $\dfrac{dy}{dx}$, but we shall find that the resulting expression still involves both variables.

The following example illustrates what is meant.

Example 4.1

Find the gradient $\dfrac{dy}{dx}$ at the point $(1, 2)$ on the curve whose equation is

$$x^3 - 5xy^2 + y^3 + 11 = 0.$$

Figure 4.1 shows that the curve is not the graph of y as a function of x. Indeed when $x = 1$ there are three possible values of y on the part of the graph

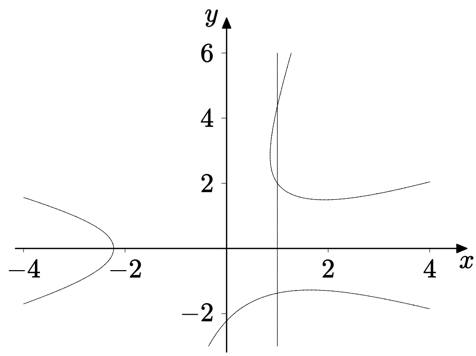

Figure 4.1 Graph of $x^3 - 5xy^2 + y^3 + 11 = 0$

shown. This is indicated by the inclusion of the line $x = 1$ in Figure 4.1. One of the three intersections of this line with the graph is of course the given point $(1, 2)$. If we consider a small part of the curve in the neighbourhood of that point then it is the graph of a function $y = y(x)$ which is one of the solutions of the equation of the curve and which specifies part of the graph near to $(1, 2)$. We cannot find $y(x)$ explicitly in terms of x, otherwise we would be able to use the normal procedures of differentiation.

The function $y(x)$ satisfies the equation of the curve, namely

$$x^3 - 5x(y(x))^2 + (y(x))^3 + 11 = 0.$$

We therefore have to use the chain rule to differentiate the y^2 and y^3 terms, and the product rule for the second term, involving x and y. Using the chain rule for the terms involving powers of $y(x)$ gives

$$\begin{aligned} \frac{d}{dx}\left(y(x)^3\right) &= 3\,(y(x))^2\,\frac{dy}{dx}; \\ \frac{d}{dx}\left(y(x)^2\right) &= 2y(x)\frac{dy}{dx}. \end{aligned}$$

Differentiating the equation of the curve with respect to x therefore gives

$$3x^2 - 5\left(x.2y(x)\frac{dy}{dx} + (y(x))^2\right) + 3\,(y(x))^2\,\frac{dy}{dx} = 0.$$

Rearranging this gives

$$\left(3(y(x))^2 - 10xy(x)\right)\frac{dy}{dx} = 5(y(x))^2 - 3x^2,$$

and therefore

$$\frac{dy}{dx} = \frac{5(y(x))^2 - 3x^2}{3(y(x))^2 - 10xy(x)} = \frac{5y^2 - 3x^2}{3y^2 - 10xy}.$$

The gradient at the point $(1,2)$ is then found by substituting these values for x and y in this expression, giving $-17/8$. This value is consistent with Figure 4.1, where the tangent line at $(1,2)$ does indeed appear to have a fairly steep negative gradient.

When we get used to this procedure we do not normally write $y(x)$ in full, but just use y throughout, as in the next example.

Example 4.2

Given $\cos(xy) = \exp(x+y)$, find $\dfrac{dy}{dx}$ in terms of x and y.

This is a purely algebraic problem. We first apply the chain rule to both sides, giving

$$-\sin(xy)\frac{d}{dx}(xy) = \exp(x+y)\frac{d}{dx}(x+y).$$

The left hand side needs the product rule, and applying this gives

$$-\sin(xy)\left(y + x\frac{dy}{dx}\right) = \exp(x+y)\left(1 + \frac{dy}{dx}\right).$$

We now collect all the terms involving the derivative and then divide to isolate the derivative, as we did in Example 4.1. We then obtain

$$\frac{dy}{dx} = -\frac{\exp(x+y) + y\sin(xy)}{x\sin(xy) + \exp(x+y)},$$

provided that the denominator is not zero.

Note that substituting arbitrary values of x and y in this equation is meaningless. The point (x,y) would have to satisfy the original equation in order that $\dfrac{dy}{dx}$ could be interpreted as the gradient of the curve.

Example 4.3

Given $xy + \mathrm{e}^y = 0$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of x and y.

Differentiating the equation with respect to x gives

$$y + x\frac{dy}{dx} + \mathrm{e}^y\frac{dy}{dx} = 0.$$

We could solve this to find the derivative, and then differentiate the resulting equation. Instead we differentiate once more without rearranging first, giving

$$\frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} + \mathrm{e}^y\frac{dy}{dx}.\frac{dy}{dx} + \mathrm{e}^y\frac{d^2y}{dx^2} = 0.$$

We now rearrange the two equations to give

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{y}{x+\mathrm{e}^y}; \\
\frac{d^2y}{dx^2} &= -\frac{2\frac{dy}{dx}+\mathrm{e}^y\left(\frac{dy}{dx}\right)^2}{x+\mathrm{e}^y} \\
&= -\frac{-\frac{2y}{x+\mathrm{e}^y}+\mathrm{e}^y\left(\frac{-y}{x+\mathrm{e}^y}\right)^2}{x+\mathrm{e}^y} \\
&= -\frac{-2y(x+\mathrm{e}^y)+y^2\mathrm{e}^y}{(x+\mathrm{e}^y)^3}.
\end{aligned}$$

One can imagine that if the initial equation were more complicated then finding the second derivative would be very involved, and so it is useful to see how MAPLE could tackle the calculations. We might think that we could undertake the first step in the calculations above using the command `diff(x*y+exp(y),x);` Unfortunately this just returns the output y, because MAPLE does not know that y is meant to be a function of x. We must use $y(x)$ in place of y, as we did in the first example in this section. The following sequence of commands can be used to solve the problem.

```
diff(x*y(x)+exp(y(x)),x);
diff(%,x);
```

We now rearrange this equation to find the second derivative, using

```
solve(%,diff(y(x),x$2));
```

We then have to substitute for dy/dx using

```
subs(diff(y(x),x)=-y(x)/(x+exp(y(x))),%);
```

and finally

```
simplify(%);
```

In the penultimate command we have typed in the expression for dy/dx, to mirror that substitution step in the algebraic process. In fact it is possible to use MAPLE to avoid having to do this, but we shall not discuss that here.

4.2 Logarithmic Differentiation

This topic is an application of implicit differentiation. It is a technique which is useful when we have expressions involving the variable in an exponent. It can also be applied to complicated products.

Example 4.4

Differentiate $y = x^{\sin x}$.

We take logarithms of both sides of the equation, to give

$$\ln y = \ln\left(x^{\sin x}\right) = \sin x.\ln x.$$

We deal with the left hand side using implicit differentiation, and the right hand side using the product rule. This gives

$$\frac{1}{y}\frac{dy}{dx} = \cos x.\ln x + \sin x.\frac{1}{x}.$$

We therefore deduce that

$$\frac{dy}{dx} = y\left(\cos x.\ln x + \sin x.\frac{1}{x}\right) = x^{\sin x}\left(\cos x.\ln x + \sin x.\frac{1}{x}\right).$$

Example 4.5

Differentiate $y = a^x$.

In fact we have already encountered this function, in Example 3.9, where we used the definition of a^x in terms of the exponential function. It is worth noting again that we cannot use the rule for differentiating powers which applies when the power is a **constant**. Using that rule would give $\dfrac{dy}{dx} = xa^{x-1}$ and this is WRONG, as is confirmed if we try to apply the rule to the exponential function. This would give the erroneous calculation

$$\frac{d}{dx}\mathrm{e}^x = x\mathrm{e}^{x-1},$$

which we know to be incorrect.

On this occasion we obtain the result by logarithmic differentiation, which gives

$$y = a^x; \quad \ln y = x\ln a; \quad \frac{1}{y}\frac{dy}{dx} = \ln a; \quad \frac{dy}{dx} = y\ln a = a^x\ln a.$$

Example 4.6

Differentiate $y = x^2\sin x\cosh x\,\mathrm{e}^x$.

We could use the product rule, but taking logarithms converts the expression into a sum, in which we can differentiate each term separately. Taking logarithms gives

$$
\begin{aligned}
\ln y &= \ln(x^2) + \ln(\sin x) + \ln(\cosh x) + \ln(\mathrm{e}^x) \\
&= 2\ln x + \ln(\sin x) + \ln(\cosh x) + x. \\
\text{Differentiating gives } \frac{1}{y}\frac{dy}{dx} &= \frac{2}{x} + \cot x + \tanh x + 1. \\
\text{Therefore } \frac{dy}{dx} &= y\left(\frac{2}{x} + \cot x + \tanh x + 1\right) \\
&= \left(x^2 \sin x \cosh x\, \mathrm{e}^x\right)\left(\frac{2}{x} + \cot x + \tanh x + 1\right). \\
&= 2x\sin x\cosh x\,\mathrm{e}^x + x^2\cos x\cosh x\,\mathrm{e}^x \\
&\quad + x^2 \sin x \sinh x\,\mathrm{e}^x + x^2\sin x\cosh x\,\mathrm{e}^x.
\end{aligned}
$$

Example 4.7

Differentiate $y = \left(x^3\mathrm{e}^x\right)^{\sin x}$.

This example combines a product with an exponent. Taking logarithms gives

$$\ln y = \sin x \ln\left(x^3\mathrm{e}^x\right) = \sin x\left(\ln x^3 + \ln \mathrm{e}^x\right) = \sin x\,(3\ln x + x).$$

Differentiating with respect to x now gives

$$\frac{1}{y}\frac{dy}{dx} = \cos x\,(3\ln x + x) + \sin x\left(\frac{3}{x} + 1\right),$$

and therefore

$$\frac{dy}{dx} = \left(x^3\mathrm{e}^x\right)^{\sin x}\left(\cos x\,(3\ln x + x) + \sin x\left(\frac{3}{x} + 1\right)\right).$$

4.3 Parametric Differentiation

Equations of curves are often given parametrically, for example the ellipse specified by

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi.$$

We want to find the gradient $\dfrac{dy}{dx}$, but the parametric equations can only be differentiated with respect to t.

We can approach this in two ways. Firstly we can use the chain rule to give

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}, \quad \text{so} \quad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}, \quad \text{provided} \quad \frac{dx}{dt} \neq 0.$$

Secondly we can go back to the limit definition of the derivative. If we want to work out the gradient $\dfrac{dy}{dx}$ at a point specified by $t = k$, we need to calculate the chord slope limit as $t \to k$. We reason as follows.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{t\to k}\frac{y(t)-y(k)}{x(t)-x(k)} = \lim_{t\to k}\frac{y(t)-y(k)}{x(t)-x(k)}\,\frac{t-k}{t-k}\\ &= \lim_{t\to k}\frac{y(t)-y(k)}{t-k}\,\frac{t-k}{x(t)-x(k)}\\ &= \lim_{t\to k}\frac{y(t)-y(k)}{t-k}\Bigg/\lim_{t\to k}\frac{x(t)-x(k)}{t-k} = \frac{dy}{dt}\Bigg/\frac{dx}{dt},\\ &\quad \text{provided } \frac{dx}{dt} \neq 0.\end{aligned}$$

Example 4.8

We shall use the formula developed above to find the gradient at an arbitrary point t on the ellipse specified by

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi.$$

The gradient is given by

$$\frac{dy}{dx} = \frac{dy}{dt}\Bigg/\frac{dx}{dt} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t.$$

The calculation is valid provided $\sin t \neq 0$, which excludes the points given by $t = 0, \pm\pi, \pm 2\pi, \ldots$, where the tangent to the ellipse is parallel to the y-axis.

Example 4.9

Find $\dfrac{dy}{dx}$ given that $x = t^2, y = t^3$.

These are the parametric equations of a curve known as a semicubical parabola. Its graph is shown in Figure 11.2, where we calculate the length of part of this curve.

The derivative is given by

$$\frac{dy}{dx} = \frac{dy}{dt}\Bigg/\frac{dx}{dt} = \frac{3t^2}{2t} = \frac{3t}{2} \quad (t \neq 0).$$

In this case we can eliminate the parameter t to give $y^2 = x^3$, and so we could also find the derivative using implicit differentiation, as follows.

$$2y\frac{dy}{dx} = 3x^2, \quad \text{so} \quad \frac{dy}{dx} = \frac{3x^2}{2y} \quad (y \neq 0).$$

We can see that the graph in Figure 11.2 is not the graph of a function, and so we would need values of both x and y to specify a point of the curve, and hence find the gradient. With the parametric form, a given value of t determines both x and y, and hence a unique point on the curve. That value of t will determine the gradient at that point.

Example 4.10

Given $x = a\cos t,\quad y = b\sin t,\quad 0 \le t \le 2\pi$, find $\dfrac{d^2y}{dx^2}$.

It is possible to find a general formula for the second derivative, but it is clearer to argue as follows.

We recall that $\dfrac{d^2y}{dx^2} = \dfrac{dY}{dx}$, where $Y = \dfrac{dy}{dx}$.

Applying the parametric differentiation formula to Y gives

$$\frac{dY}{dx} = \frac{dY}{dt}\Bigg/\frac{dx}{dt}.$$

We worked out Y in Example 4.8, and so we apply this formula, giving

$$\frac{d^2y}{dx^2} = \frac{dY}{dx} = \frac{dY}{dt}\Bigg/\frac{dx}{dt} = -\frac{b}{a}\left(\operatorname{cosec}^2 t\right)/(-a\sin t) = -\frac{b}{a^2\sin^3 t},$$

provided $\sin t \neq 0$.

NOTE: A common mistake is to try to find $\dfrac{d^2y}{dx^2}$ by differentiating the formula obtained for $\dfrac{dy}{dx}$ with respect to t. This is WRONG.

4.4 Differentiating Inverse Functions

Inverse functions were discussed in some detail in Section 1.7, and we now consider their differentiation. It is possible to find a general formula, as we shall demonstrate, but in most cases it is more helpful to use an implicit function approach, and this is done in the examples in this section.

Suppose that we have a differentiable function f with its inverse g. So $y = f(x)$ and $x = g(y)$ are equivalent. We shall establish differentiability of g using the limit definition.

$$\frac{dg}{dy} = \lim_{k\to 0}\frac{g(y+k) - g(y)}{k}.$$

Now $y+k=f(x+h)$ for some h, and since f is continuous it follows that $k\to 0$ as $h\to 0$. Also, since f has an inverse, it is 1-1, so for $h\neq 0$ we have $f(x+h)\neq f(x)$, so that $k\neq 0$. Therefore

$$\frac{g(y+k)-g(y)}{k}=\frac{g(y+k)-g(y)}{y+k-y}=\frac{x+h-x}{f(x+h)-f(x)}=\frac{h}{f(x+h)-f(x)}.$$

From this we deduce that

$$\frac{dg}{dy}=\lim_{k\to 0}\frac{g(y+k)-g(y)}{k}=\lim_{h\to 0}\frac{h}{f(x+h)-f(x)}=1\Big/\frac{df}{dx}\,.$$

If we were to assume that the inverse g is differentiable then we could obtain the same formula from the inverse function relationship $g(f(x))=x$. Differentiating this equation using the chain rule gives $g'(f(x))f'(x)=1$, and therefore, since $y=f(x)$,

$$g'(y)=\frac{1}{f'(x)}.$$

Example 4.11

We can verify the above rule using the logarithmic function.

Suppose $y=f(x)=\ln x$, so that $x=g(y)=\mathrm{e}^y$ is the inverse. then

$$g'(y)=\mathrm{e}^y=\mathrm{e}^{\ln x}=x=\frac{1}{\frac{1}{x}}=\frac{1}{f'(x)}.$$

Example 4.12

Find the derivative of $\sinh^{-1}x$.

Suppose $y=\sinh^{-1}x$, so that $x=\sinh y$. Differentiating the latter equation implicitly with respect to x gives

$$1=\cosh y\frac{dy}{dx}\quad\text{so that}\quad\frac{dy}{dx}=\frac{1}{\cosh y},$$

as the general formula above implies. However we want the answer in terms of x, and so we have to find $\cosh y$ in terms of $x=\sinh y$. Using the hyperbolic identity $\cosh^2 y-\sinh^2 y=1$ gives $\cosh y=\sqrt{1+\sinh^2 y}$, where we use the positive square root because $\cosh y$ is always positive. Therefore

$$\frac{dy}{dx}=\frac{1}{\cosh y}=\frac{1}{\sqrt{1+\sinh^2 y}}=\frac{1}{\sqrt{1+x^2}}.$$

Example 4.13

Let $f(x) = x^3 + 2x - 2$, which is a 1-1 function. Find the derivative of $f^{-1}(x)$ at the point where f and its inverse intersect.

The graphs of the function and its inverse are shown in Figure 4.2.

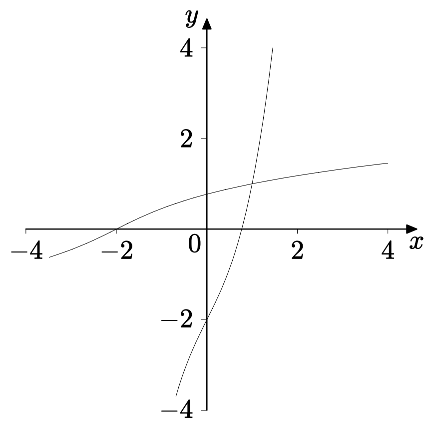

Figure 4.2 Graph of $x^3 + 2x - 2$ and its inverse

The two graphs intersect at the point $(1, 1)$, as can be seen from the fact that $f(1) = 1$. Calculating the derivative gives $f'(x) = 3x^2 + 2$, and so $f'(1) = 5$. Therefore at the point of intersection the derivative of the inverse function has value $1/5$.

Example 4.14

In this example we consider the problem of differentiating the inverse sine function. In Section 1.7.2 we considered the problems involved in restricting the domain of sine so as to obtain a 1-1 function, which would therefore have an inverse. We have to consider the same approach here.

Suppose that $y = \sin^{-1} x$, which is equivalent to $x = \sin y$. Differentiating the latter equation implicitly with respect to x gives

$$1 = \cos y \frac{dy}{dx} \quad \text{so that} \quad \frac{dy}{dx} = \frac{1}{\cos y}.$$

We want the result in terms of x, and so we use the identity $\cos^2 y + \sin^2 y = 1$, giving $\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}$. Unlike the previous example, where sinh is 1-1 over its whole domain, and where the choice of square root was straightforward, in this case we have to consider how the domain is restricted in the same way as in Section 1.7.2. Recall that in that section we used the notation $\sin^{-1} x$ to denote the inverse of the function specified by

$$f(x) = \sin x; \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

The graph was shown in Figure 1.22, and we can see that the gradient of the inverse is positive, which means we have to choose the positive square root. We note also that the derivative of sin is cos, which itself is positive in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, again confirming the choice of the positive square root. So with f specified with the given domain we have

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}.$$

If on the other hand we consider the function g specified by

$$g(x) = \sin x; \quad \frac{\pi}{2} \leq x \leq \frac{3\pi}{2},$$

then the gradient of the inverse is negative, as shown in Figure 1.23, and also confirmed by the fact that cosine is negative in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. So in this case we have

$$\frac{d}{dx}\left(g^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}.$$

4.5 Leibniz Theorem

We saw in Example 3.14 that finding the n-th derivative of a product can be complicated. In this section we derive a general formula for this procedure. If we begin by applying the product rule three times to the general expression of the form $h(x) = f(x)g(x)$ and collect like terms together at each stage we soon perceive a pattern emerging. We find that

$$\begin{aligned}
h'(x) &= f'(x)g(x) + f(x)g'(x); \\
h''(x) &= [f''(x)g(x) + f'(x)g'(x)] + [f'(x)g'(x) + f(x)g''(x)] \\
&= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x); \\
h'''(x) &= [f'''(x)g(x) + f''(x)g'(x)] + 2[f''(x)g'(x) + f'(x)g''(x)] \\
&\quad +[f'(x)g''(x) + f(x)g'''(x)] \\
&= f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x),
\end{aligned}$$

where at each stage the square brackets indicate a pair of terms arising from the application of the product rule to a single term at the previous stage.

The pattern of numerical coefficients when the terms are collected together is that of the binomial coefficients from Pascal's triangle, and this enables us to formulate the general result.

Theorem 4.15 (Leibniz)

If the functions $f(x), g(x)$ are both differentiable n times then their product is differentiable n times and

$$\begin{aligned}\frac{d^n}{dx^n}(fg) &= \frac{d^n f}{dx^n}g + \binom{n}{1}\frac{d^{n-1}f}{dx^{n-1}}\frac{dg}{dx} + \binom{n}{2}\frac{d^{n-2}f}{dx^{n-2}}\frac{d^2g}{dx^2} + \ldots \\ &+ \binom{n}{k-1}\frac{d^{n-(k-1)}f}{dx^{n-(k-1)}}\frac{d^{k-1}g}{dx^{k-1}} + \binom{n}{k}\frac{d^{n-k}f}{dx^{n-k}}\frac{d^kg}{dx^k} + \ldots + f\frac{d^ng}{dx^n} \\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{d^{n-k}f}{dx^{n-k}}\frac{d^kg}{dx^k}.\end{aligned}$$

Proof

The pattern we established above provides evidence for the truth of the result. We include a proof here for readers who are familiar with proof by induction (see Howie, Chapter 1) and the basic properties of binomial coefficients

In the course of the proof we use relationships involving binomial coefficients, which we prove first. It may be helpful to remind readers of the definition and notation for binomial coefficients. They occur in binomial expansions such as $(1+x)^n$, where n is a positive integer. The k-th binomial coefficient is the coefficient of x^k in this expansion. It is given by the following formula

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

for $k = 0, 1, \ldots, n$, where $0!$ is defined to be 1. In fact for $k = 0$ and $k = n$ respectively we have

$$\binom{n}{0} = \frac{n!}{(n-0)!0!} = 1, \quad \binom{n}{n} = \frac{n!}{(n-n)!n!} = 1.$$

These expressions occur in Pascal's Triangle, and the following addition rule expresses in general terms the way in which we obtain the coefficients in a particular row by adding the two appropriate entries from the row above.

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!}$$

$$
\begin{aligned}
&= \frac{n!}{(n-k+1)!k!}\left(k+(n-k+1)\right) \\
&= \frac{(n+1)!}{(n+1-k)!k!} = \binom{n+1}{k}.
\end{aligned}
$$

We note the particular case $k = 1$, which we use below. This states that

$$
\binom{n}{0} + \binom{n}{1} = \binom{n+1}{1}, \quad \text{i.e.,} \quad 1 + \binom{n}{1} = \binom{n+1}{1}.
$$

The proof of Leibniz Theorem uses the method of mathematical induction. The result is true for $n = 1$ because it is just the ordinary product rule. If the result is true for n as in the statement of the theorem then we differentiate both sides once more with respect to x. Each term on the right hand side gives rise to two terms, from the product rule. We therefore have

$$
\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}}(fg) &= \frac{d^{n+1}f}{dx^{n+1}}g + \frac{d^n f}{dx^n}\frac{dg}{dx} \\
&+ \binom{n}{1}\frac{d^n f}{dx^n}\frac{dg}{dx} + \binom{n}{1}\frac{d^{n-1}f}{dx^{n-1}}\frac{d^2 g}{dx^2} \\
&+ \binom{n}{2}\frac{d^{n-1}f}{dx^{n-1}}\frac{d^2 g}{dx^2} + \binom{n}{2}\frac{d^{n-2}f}{dx^{n-2}}\frac{d^3 f}{dx^3} + \cdots \\
&+ \binom{n}{k-1}\frac{d^{n-k+2}f}{dx^{n-k+2}}\frac{d^{k-1}g}{dx^{k-1}} + \binom{n}{k-1}\frac{d^{n-k+1}f}{dx^{n-k+1}}\frac{d^k g}{dx^k} \\
&+ \binom{n}{k}\frac{d^{n-k+1}f}{dx^{n-k+1}}\frac{d^k g}{dx^k} + \binom{n}{k}\frac{d^{n-k}f}{dx^{n-k}}\frac{d^{k+1}g}{dx^{k+1}} + \cdots \\
&+ \frac{df}{dx}\frac{d^n g}{dx^n} + f\frac{d^{n+1}g}{dx^{n+1}}.
\end{aligned}
$$

We now rearrange the terms in pairs so that they contain the same derivative. So the second term in line 1 of the above chain of expressions combines with the first term in line 2, the second term in line 2 with the first term in line 3, and so on. This now gives

$$
\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}}(fg) &= \frac{d^{n+1}f}{dx^{n+1}}g \\
&+ \frac{d^n f}{dx^n}\frac{dg}{dx} + \binom{n}{1}\frac{d^n f}{dx^n}\frac{dg}{dx} \\
&+ \binom{n}{1}\frac{d^{n-1}f}{dx^{n-1}}\frac{d^2 g}{dx^2} + \binom{n}{2}\frac{d^{n-1}f}{dx^{n-1}}\frac{d^2 g}{dx^2} \\
&+ \binom{n}{2}\frac{d^{n-2}f}{dx^{n-2}}\frac{d^3 f}{dx^3} + \cdots
\end{aligned}
$$

$$\begin{aligned}
&+ \binom{n}{k-1}\frac{d^{n-k+2}f}{dx^{n-k+2}}\frac{d^{k-1}g}{dx^{k-1}} \\
&+ \binom{n}{k-1}\frac{d^{n-k+1}f}{dx^{n-k+1}}\frac{d^{k}g}{dx^{k}} + \binom{n}{k}\frac{d^{n-k+1}f}{dx^{n-k+1}}\frac{d^{k}g}{dx^{k}} \\
&+ \binom{n}{k}\frac{d^{n-k}f}{dx^{n-k}}\frac{d^{k+1}g}{dx^{k+1}} + \cdots \\
&+ \frac{df}{dx}\frac{d^{n}g}{dx^{n}} + f\frac{d^{n+1}g}{dx^{n+1}}.
\end{aligned}$$

Finally we utilise the addition rule for binomial coefficients to give

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}}(fg) &= \frac{d^{n+1}f}{dx^{n+1}}g + \binom{n+1}{1}\frac{d^{n+1-1}f}{dx^{n+1-1}}\frac{dg}{dx} \\
&+ \binom{n+1}{2}\frac{d^{n+1-2}f}{dx^{n+1-2}}\frac{d^{2}g}{dx^{2}} + \cdots \\
&+ \binom{n+1}{\mathrm{k}}\frac{d^{n+1-k}f}{dx^{n+1-k}}\frac{d^{k}g}{dx^{k}} + \cdots + f\frac{d^{n+1}g}{dx^{n+1}},
\end{aligned}$$

which is the result for $n+1$, thereby completing the proof by induction.

□

Example 4.16

Find a formula for the n-th derivative of $x^2 \ln(2x+3)$.

Let $f(x) = \ln(2x+3)$; $g(x) = x^2$. Notice that for g the third and subsequent derivatives are all zero, so that only the first three terms in Leibniz formula are non-zero. We use the formula for the n-th derivative of $f(x)$ which we obtained in Example 3.13, namely

$$f^{(n)}(x) = (n-1)!\frac{(-1)^{(n+1)}2^n}{(2x+3)^n}.$$

Leibniz Theorem therefore gives

$$\begin{aligned}
(fg)^{(n)}(x) &= f^{(n)}(x).x^2 + \binom{n}{1}f^{(n-1)}(x).2x + \binom{n}{2}f^{(n-2)}.2 \\
&= (n-1)!\frac{(-1)^{n+1}2^n}{(2x+3)^n}.x^2 \\
&+ n.(n-2)!\frac{(-1)^{n}2^{n-1}}{(2x+3)^{n-1}}.2x \\
&+ \frac{n(n-1)}{2!}.(n-3)!\frac{(-1)^{n-1}2^{n-2}}{(2x+3)^{n-2}}.2.
\end{aligned}$$

If we take out the factor of $\dfrac{(n-3)!(-1)^{(n+1)}2^{n-2}}{(2x+3)^n}$ from each of the three terms above we are left with

$$(n-1)(n-2).2^2.x^2 - n(n-2).2.2x(2x+3) + n(n-1)(2x+3)^2,$$

which simplifies to $8x^2 + 12nx + 9n^2 - 9n$. We have therefore shown that

$$\frac{d^n}{dx^n}\left(\ln(2x+3)\right) = \frac{(n-3)!(-1)^{(n+1)}2^{n-2}}{(2x+3)^n}\left(8x^2 + 12nx + 9n^2 - 9n\right).$$

EXERCISES

4.1. For each of the following, find $\dfrac{dy}{dx}$ in terms of x and y.

(a) $2xy + x - 3y = 2$; (b) $(x+1)^2 + 2(y-1)^3 = 0$;

(c) $x^3y^3 = xy - 1$; (d) $y\ln x = x\ln y$;

(e) $x^2 - 3xy^2 + y^3x - y^2 = 2$; (f) $xy\sqrt{x+y} = 1$;

(g) $\dfrac{x}{y} - \dfrac{y}{x} = 1$; (h) $\dfrac{x+y}{x-y} = \dfrac{x}{y} + \dfrac{1}{y^2}$;

(i) $\sin(xy^2) = x + \cos(yx^2)$; (j) $xy\exp\left(\dfrac{x}{y}\right) = 1$.

4.2. For each of the following, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of x and y.

(a) $xy = 2x - 3y$; (b) $x\sin y = \sin x$.

4.3. Given that $x^2 + 2y^2 = 4$, find $\dfrac{d^2y}{dx^2}$ in terms of y only.

4.4. Find the gradient at the point $(2,-2)$ of the curve whose equation is $x^3 - xy - 3y^2 = 0$. Hence determine the equation of the tangent to the curve at that point.

4.5. Find the gradient at the point $(1,0)$ of the curve whose equation is $x\sin(xy) = x^2 - 1$. Hence determine the equation of the tangent to the curve at that point. Explain from the formula why the curve is symmetric about the y-axis, and hence write down the equation of the tangent at the point $(-1,0)$.

Verify the symmetry by plotting the graph using MAPLE.

4.6. For each of the following, use logarithmic differentiation to find $\dfrac{dy}{dx}$, expressing the results in terms of x.

(a) $y = x^x$; (b) $y = x^{-x}$;

(c) $y = (-x)^{-x}$; (d) $y = (\sin x)^{\sin x}$;

(e) $y = (\mathrm{e}^x)^{\ln x}$; (f) $y = (\ln x)^x$;

(g) $y = (\tan x)^{2x}$; (h) $y = 2^{x+x^2}$;

(i) $y = \sqrt{\mathrm{e}^x \sin x}$; (j) $y = \sqrt{(x-1)^2 \mathrm{e}^{-x} \cos x}$.

4.7. For each of the following, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t. Plot each of the curves using MAPLE.

(a) $x = 1 + \ln t,\ y = t^2 - t$; (b) $x = t + t^2,\ y = t - t^2$;

(c) $x = t \ln t,\ y = 2t + 3$; (d) $x = t^2,\ y = \mathrm{e}^t + 1$;

(e) $x = t^3,\ y = \sqrt{t^2+1}$; (f) $x = \sin(t^2),\ y = \cos t$;

(g) $x = \tan t,\ y = \mathrm{e}^t$; (h) $x = \ln(\cos t),\ y = \sin t$;

(i) $x = \cos t,\ y = t\mathrm{e}^t$; (j) $x = \sin(t),\ y = t^2 + 1$.

4.8. Find an expression for the gradient $\dfrac{dy}{dx}$, in terms of t, at a point on the hyperbola given by

$$x = a \cosh t, \quad y = b \sinh t.$$

Write down any values of t for which the gradient is undefined, explaining the geometrical significance.

Find expressions for $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^3y}{dx^3}$, in terms of t.

4.9. Find an expression, in terms of t, for the gradient at a point on the curve specified by

$$x = t \cos t, \quad y = t \sin t, \quad t \geq 0.$$

Plot the curve using MAPLE and explain why there are infinitely many values of t for which the gradient is undefined.

4.10. For each of the following functions, none of which is 1-1, investigate differentiation of inverse functions obtained by restricting the domain in various ways, as in Example 4.14.

$$\text{(a)}\quad f(x) = \cosh x; \quad \text{(b)}\quad f(x) = \tan x; \quad \text{(c)}\quad f(x) = \mathrm{e}^{x^2}.$$

4.11. Find an expression for the derivative of $\tanh^{-1} x$.

4.12. Use Leibniz Theorem to find the n-th derivative of each of the following.

$$\text{(a)}\quad x \ln x; \quad \text{(b)}\quad (x^2 - 2x + 3)\mathrm{e}^{2x}; \quad \text{(c)}\quad x^3\mathrm{e}^{-x}.$$