Chapter 1

Systems of Linear Equations

1.1 Intro. to systems of linear equations

Homework: [Textbook, Ex. 13, 15, 41, 47, 49, 51, 73; page 10-].

Main points in this section:

1. *Definition of Linear system of equations and homogeneous systems.*

2. *Row-echelon form of a linear system and Gaussian elimination.*

3. *Solving linear system of equations using Gaussian elimination.*

Definition 1.1.1. A linear equation in n (unknown) variables $x_1, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Here $a_1, a_2, \ldots, a_n, b$ are real numbers. We say b is the constant term and a_i is the coefficient of x_i.

For real numbers $s_1, \ldots, s_n$, if

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

we say that

$$x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$$

is a solution of this equation.

1. An example of a linear equation in two unknowns is $2x + 7y = 5$. A solution of this equation is $x = -1, y = 1$. The equation has many more solutions. **The graph of this equation is a line.**

2. An example of a linear equation in three unknowns is $2x + y + \pi z = \pi$. A solution of this equation is $x = 0, y = 0, z = 1$. The equation has many more solutions. **The graph of this equation (in 3-space) is a plane.**

3. See [Textbook, Example 1, page 2] for examples of linear and non-linear equations.

Definition 1.1.2. By a **System of Linear Equations** in n variables $x_1, x_2, \ldots, x_n$ we mean a collection of linear equations in these variables. A system of m linear equations in these n variables can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \cdots \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where a_{ij} and b_i are all real numbers.

Such a linear system is called a **homogeneous linear system** if

$$b_1 = b_2 = \cdots = b_m = 0.$$

1. A solution to such a system is a sequence of n numbers $s_1, \ldots, s_n$ that is solution to all these m equations.

2. In two variables, here is an example of a system of two equation:

$$\begin{cases} 2x + y = 3 \\ x - 9y = -8 \end{cases}$$

   Clearly, $x = 1, y = 1$ is the (only) solution to this system.

   Geometrically, solution given by precisely the point where the graphs (two lines) of these two equations meet.

   Also note that the system

$$\begin{cases} 2x + y = 3 \\ 2x + y = 7 \end{cases}$$

   does not have any solution. Such a system would be called an **inconsistent** system. *Geometrically, these two equations in the system represent two parallel lines (they never meet).*

3. In three variables, the following is an example of a system of two equation:

$$\begin{cases} 2x + y + 2z = 3 \\ x - 9y + 2z = -8 \end{cases}$$

   Clearly, $x = 1, y = 1, z = 0$ is a solution to this system. This system has many more solutions. For example,

$$x = 11, y = 0, z = -19/2$$

   is also a solution of this system. *Geometrically, solution given by precisely the points where the graphs (two planes) in 3-space of these two equations meet.*

4. **Classification of linear systems:** Given a linear system in n variables, precisely on the the following three is true:

   (a) The system has exactly one solution (consistent system).

   (b) The system has infinitely many solutions (consistent system).

   (c) The system has NO solution (inconsistent system).

5. Two systems of linear equations are called **equivalent**,
   if they have precisely the same set of solutions.

6. Following operations on a system produces an equivalent system:

   (a) Interchange two equations.

   (b) Multiply an equation by a nonzero constant.

   (c) Add a multiple of an equation to another one.

   These three operations are sometimes known as
   basic or elementary operations.

7. A linear system of the form

$$\begin{cases} x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ \qquad\qquad x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\qquad\qquad x_3 + \cdots + a_{3n}x_n = b_3 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots \end{cases}$$

is said to be in **row-echelon form** (see page 6). The point is:

(a) you drop one variable in each successive equation (step),

(b) The coefficient of the "leading variable" in equation is 1.

In two variables x, y this would (sometime) look like

$$\begin{cases} x + a_{12}y = b_1 \\ \qquad\quad y = b_2 \end{cases}$$

In three variables x, y, z this would (sometime) look like

$$\begin{cases} x + a_{12}y + a_{13}z = b_1 \\ \qquad\quad y + a_{23}z = b_2 \\ \qquad\qquad\qquad z = b_3 \end{cases}$$

Theorem 1.1.3. The following are some facts:

1. Any system of linear equations is equivalent to a linear system in row-echelon form.
2. This can be achieved by a sequence of application of the three basic elementary operation described in (6).
3. This process is known as **Gaussian elimination**.

Read Examples 5-9 (page 6-).

Practice: For exercise 31-56 (page 10), reduce the system to a row-echelon form and solve.

Exercise 1.1.4 (Ex. 32. p 11)**.** Reduce the following system and solve:

$$\begin{cases} 4x - 5y = 3 & Eqn-1 \\ -8x + 10y = 14 & Eqn-2 \end{cases}$$

Add 2 times Enq-1 to Eqn-2:

$$\begin{cases} 4x - 5y = 3 & Eqn-1 \\ 0 = 20 & Eqn-3 \end{cases}$$

The Eqn-3 is absurd. So, the system has no solution. The system is inconsistent.

Exercise 1.1.5 (Ex. 34. p 11)**.** Reduce the following system and solve:

$$\begin{cases} 9x - 4y = 5 & Eqn-1 \\ \frac{1}{2}x + \frac{1}{3}y = 0 & Eqn-2 \end{cases}$$

Multiply Eqn-2 by 18:

$$\begin{cases} 9x - 4y = 5 & Eqn-1 \\ 9x + 6y = 0 & Eqn-3 \end{cases}$$

Subtract Eqn-1 from the Eqn-3

$$\begin{cases} 9x - 4y = 5 & Eqn-1 \\ 10y = -5 & Eqn-4 \end{cases}$$

Divide Eqn-4 by 10:

$$\begin{cases} 9x - 4y = 5 & Eqn-1 \\ y = -\frac{1}{2} & Eqn-5 \end{cases}$$

Divide Eqn-1 by 9:

$$\begin{cases} x - \frac{4}{9}y = \frac{5}{9} & Eqn-6 \\ y = -\frac{1}{2} & Eqn-5 \end{cases}$$

This is the row-echelon form.

Now substitute $y = -\frac{1}{2}$ in Eqn-6

$$x - \frac{4}{9}\left(-\frac{1}{2}\right) = \frac{5}{9} \qquad or \qquad x = \frac{1}{3}.$$

So, the solution is

$$x = \frac{1}{3}, \quad y = -\frac{1}{2}.$$

Exercise 1.1.6 (Ex. 44. p 12)**.**

$$\begin{cases} \frac{x_1+3}{4} + \frac{x_2-1}{3} = 1 & Eqn-1 \\ 2x_1 - x_2 = 12 & Eqn-2 \end{cases}$$

multiply Eqn-1 by 12 and simplify:

$$\begin{cases} 3x_1 + 4x_2 = 7 & Eqn-3 \\ 2x_1 - x_2 = 12 & Eqn-2 \end{cases}$$

Add $-\frac{2}{3}$ times Eqn-3 to Eqn-2:

$$\begin{cases} 3x_1 + 4x_2 = 7 & Eqn-3 \\ \frac{-11}{3}x_2 = \frac{22}{3} & Eqn-4 \end{cases}$$

Multiply Eqn-4 by $\frac{-3}{11}$:

$$\begin{cases} 3x_1 + 4x_2 = 7 & Eqn-3 \\ x_2 = -2 & Eqn-5 \end{cases}$$

Multiply Eqn-3 by $\frac{1}{3}$:

$$\begin{cases} x_1 + \frac{4}{3}x_2 = \frac{7}{3} & Eqn-6 \\ x_2 = -2 & Eqn-5 \end{cases}$$

So, above is the row-echelon form of the system. Now substitute $x_2 = -2$ in Eqn-6 and get $x_1 = \frac{8}{3} + \frac{7}{3} = 5$. So, the systme is consistentand has unique solution

$$x_1 = 5, \quad x_2 = -2.$$

Exercise 1.1.7 (Ex. 50, p 12)**.** Deduce an equivalent row-echelon form and solove the following system:

$$\begin{cases} 5x_1 - 3x_2 + 2x_3 = 3 & Eqn-1 \\ 2x_1 + 4x_2 - x_3 = 7 & Eqn-2 \\ x_1 - 11x_2 + 4x_3 = 3 & Eqn-3 \end{cases}$$

First, switch Eqn-1 and Eqn-3:

$$\begin{cases} x_1 - 11x_2 + 4x_3 = 3 & Eqn-3 \\ 2x_1 + 4x_2 - x_3 = 7 & Eqn-2 \\ 5x_1 - 3x_2 + 2x_3 = 3 & Eqn-1 \end{cases}$$

Subtract 2 times Eqn-3 from Eqn-2 and 5 times Eqn-3 from Eqn-1:

$$\begin{cases} x_1 - 11x_2 + 4x_3 = 3 & Eqn-3 \\ 26x_2 - 9x_3 = 1 & Eqn-4 \\ 52x_2 - 18x_3 = -12 & Eqn-5 \end{cases}$$

Subtract 2 times Eqn-4 from Eqn-5:

$$\begin{cases} x_1 - 11x_2 + 4x_3 = 3 & Eqn-3 \\ 26x_2 - 9x_3 = 1 & Eqn-4 \\ 0 = -14 & Eqn-6 \end{cases}$$

The system is inconsistent because Eqn-6 is absurd. To obtain the row-echelon form, we divid Eqn-4 by 26:

$$\begin{cases} x_1 - 11x_2 + 4x_3 = 3 & Eqn-3 \\ x_2 - \frac{9}{26}x_3 = \frac{1}{26} & Eqn-7 \\ 0 = -14 & Eqn-6 \end{cases}$$

Exercise 1.1.8 (Ex. 52, p 12)**.** Deduce an equivalent row-echelon form and solove the following system:

$$\begin{cases} x_1 \qquad + 4x_3 = 13 & Eqn-1 \\ 4x_1 - 2x_2 \quad + x_3 = 7 & Eqn-2 \\ 2x_1 - 2x_2 - 7x_3 = -19 & Eqn-3 \end{cases}$$

Subtract 4 times Eqn-1 from Eqn-2 and suntract 2 times Eqn-1 from Equn-3:

$$\begin{cases} x_1 \qquad + 4x_3 = 13 & Eqn-1 \\ -2x_2 - 15x_3 = -45 & Eqn-4 \\ -2x_2 - 15x_3 = -45 & Eqn-5 \end{cases}$$

Subtract Eqn-4 from Eqn-5:

$$\begin{cases} x_1 \qquad + 4x_3 = 13 & Eqn-1 \\ -2x_2 - 15x_3 = -45 & Eqn-4 \\ 0 = 0 & Eqn-6 \end{cases}$$

Multiply Eqn-4 by -.5 and we get

$$\begin{cases} x_1 \qquad + 4x_3 = 13 & Eqn-1 \\ x_2 + 7.5x_3 = 22.5 & Eqn-7 \\ 0 = 0 & Eqn-6 \end{cases}$$

The above is the row-echelon form of the system. The system is consistent. Since the echelon form has actually two equations and number of variables is three, the system has infinitely many solutions. For any value $x_3 = t$, we have

$$x_2 = 22.5 - 7.5t \qquad and \qquad x_1 = 13 - 4t.$$

So, a parametric solution of this system is

$$x_1 = 13 - 4t, \qquad x_2 = 22.5 - 7.5t, \qquad x_3 = t.$$

Exercise 1.1.9 (Ex. 56, p 12)**.** Deduce an equivalent row-echelon form and solve the following system:

$$\left\{\begin{array}{rrrrrl} x_1 & & & +3x_4 & = 4 & \qquad Eqn-1 \\ & 2x_2 & -x_3 & -x_4 & = 0 & \qquad Eqn-2 \\ & 3x_2 & & -2x_4 & = 1 & \qquad Eqn-3 \\ 2x_1 & -x_2 & +4x_3 & & = 5 & \qquad Eqn-4 \end{array}\right.$$

Subtract 2 time Eqn-1 from Eqn-4:

$$\left\{\begin{array}{rrrrrl} x_1 & & & +3x_4 & = 4 & \qquad Eqn-1 \\ & 2x_2 & -x_3 & -x_4 & = 0 & \qquad Eqn-2 \\ & 3x_2 & & -2x_4 & = 1 & \qquad Eqn-3 \\ & -x_2 & +4x_3 & -6x_4 & = -3 & \qquad Eqn-5 \end{array}\right.$$

Multiply Eqn-2 by .5:

$$\left\{\begin{array}{rrrrrl} x_1 & & & +3x_4 & = 4 & \qquad Eqn-1 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 & \qquad Eqn-6 \\ & 3x_2 & & -2x_4 & = 1 & \qquad Eqn-3 \\ & -x_2 & +4x_3 & -6x_4 & = -3 & \qquad Eqn-5 \end{array}\right.$$

Subtract 3 times Eqn-6 from Eqn-3 and add Eqn-2 to Eqn-5:

$$\left\{\begin{array}{rrrrrl} x_1 & & & +3x_4 & = 4 & \qquad Eqn-1 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 & \qquad Eqn-6 \\ & & 1.5x_3 & -.5x_4 & = 1 & \qquad Eqn-7 \\ & & 3.5x_3 & -6.5x_4 & = -3 & \qquad Eqn-8 \end{array}\right.$$

Myltiply Eqn-7 by $\frac{2}{3}$:

$$\begin{cases} x_1 & & & +3x_4 & = 4 & Eqn-1 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 & Eqn-6 \\ & & x_3 & -\frac{1}{3}x_4 & = \frac{2}{3} & Eqn-9 \\ & & 3.5x_3 & -6.5x_4 & = -3 & Eqn-8 \end{cases}$$

Subtract 3.5 times Eqn-9 from Eqn-8:

$$\begin{cases} x_1 & & & +3x_4 & = 4 & Eqn-1 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 & Eqn-6 \\ & & x_3 & -\frac{1}{3}x_4 & = \frac{2}{3} & Eqn-9 \\ & & & -\frac{16}{3}x_4 & = -\frac{16}{3} & Eqn-10 \end{cases}$$

Multiply Eqn-10 by $-\frac{3}{16}$:

$$\begin{cases} x_1 & & & +3x_4 & = 4 & Eqn-1 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 & Eqn-6 \\ & & x_3 & -\frac{1}{3}x_4 & = \frac{2}{3} & Eqn-9 \\ & & & x_4 & = 1 & Eqn-11 \end{cases}$$

The above is a row-echelon form of the system. By back-substitution:

$$x_4 = 1, \quad x_3 = \frac{2}{3} + \frac{1}{3} = 1, \quad x_2 = 1, \quad x_1 = 1.$$

1.2 Gaussian, Gauss-Jordan Elimination

Homework: §1.2, page 22-:
Ex 29, 33, 35, 47b (help), 49 (help), 53, 63 (help)

Main points in this section:

1. *Definition of matrices.*
2. *Elementary row operation on a matrix.*
3. *Definition of Row-echelon form of matrix.*
4. *Gaussian and Gauss-Jordan elimination.*
5. *Solving sytems of linear equations using Gaussian elimination and Gauss-Jordan elimination.*

Definition 1.2.1. For two positive integers m, n and $m \times n-$matrix is a rectangular array

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

1. the array has m rows ana n column.

2. Here a_{ij} is a real number, to be called $ij^{th}-$entry. This entry sits in the $i^{th}-$row $j^{th}-$column. The first subscript i of a_{ij} is called the row subscript and j is called the column subscript.

3. It is possible to talk about matrices whose entries a_{ij} are not real numbers. We can talk about matrices of any kind of objects. For example, we can consider matrices complex numbers. However, in this course, we consider matrices with real entries ONLY, and such matrices are also called **real matrices**.

4. We say that the **size** of the above matrix is $m \times n$.

5. A **square matrix** of order n is a matrix whose number of rows and columns are same and equal to n.

6. For a square matrix of order n, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the **main diaginal entries**.

The most common use, for this class, of matrices is to represent system of liner equation. Given a system of liner equations, an associated matrix to be called the **augmented matrix** contains all the information regarding the system.

Definition 1.2.2. Given a system of m linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{cases}$$

the **augmented matrix** of the system is defined as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{pmatrix}$$

and the **coefficient matrix** is defined as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}.$$

1. Conversely, given a $m \times (n+1)$ matrix, we can write down a system of linear m equations in n unknowns (variables).

2. Consider the linear system (from exercise 1.1.4):

$$\begin{cases} 4x - 5y = 3 \\ -8x + 10y = 14 \end{cases}$$

The augmented matrix of the system is

$$\begin{pmatrix} 4 & -5 & 3 \\ -8 & 10 & 14 \end{pmatrix}$$

and the coefficient matrix is

$$\begin{pmatrix} 4 & -5 \\ -8 & 10 \end{pmatrix}$$

3. Consider the linear system (from exercise 1.1.7):

$$\begin{cases} 5x_1 - 3x_2 + 2x_3 = 3 \\ 2x_1 + 4x_2 - x_3 = 7 \\ x_1 - 11x_2 + 4x_3 = 3 \end{cases}$$

The augmented and the coefficient matrices of this system are:

$$\begin{pmatrix} 5 & -3 & 2 & 3 \\ 2 & 4 & -1 & 7 \\ 1 & -11 & 4 & 3 \end{pmatrix}; \qquad \begin{pmatrix} 5 & -3 & 2 \\ 2 & 4 & -1 \\ 1 & -11 & 4 \end{pmatrix}.$$

Recall that we deduced an equivalent system in row-echelon form:

$$\begin{cases} x_1 - 11x_2 + 4x_3 = 3 & Eqn-3 \\ x_2 - \frac{9}{26}x_3 = \frac{1}{26} & Eqn-7 \\ 0 = -14 & Eqn-6 \end{cases}$$

The augmented and the coefficient of this row-echelon form is:

$$\begin{pmatrix} 1 & -11 & 4 & 3 \\ 0 & 1 & -\frac{9}{26} & \frac{1}{26} \\ 0 & 0 & 0 & -14 \end{pmatrix}; \qquad \begin{pmatrix} 1 & -11 & 4 \\ 0 & 1 & -\frac{9}{26} \\ 0 & 0 & 0 \end{pmatrix}.$$

4. Consider the linear system (from exercise 1.1.9):

$$\begin{cases} x_1 & & & +3x_4 & = 4 & Eqn-1 \\ & 2x_2 & -x_3 & -x_4 & = 0 & Eqn-2 \\ & 3x_2 & & -2x_4 & = 1 & Eqn-3 \\ 2x_1 & -x_2 & +4x_3 & & = 5 & Eqn-4 \end{cases}$$

The augmented and the coefficient matrices are:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 2 & -1 & 4 & 0 & 5 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 2 & -1 & -1 \\ 0 & 3 & 0 & -2 \\ 2 & -1 & 4 & 0 \end{pmatrix}.$$

Recall that we deduced an equivalent system in row-echelon form:

$$\begin{cases} x_1 & & & +3x_4 & = 4 & \qquad Eqn-1 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 & \qquad Eqn-6 \\ & & x_3 & -\frac{1}{3}x_4 & = \frac{2}{3} & \qquad Eqn-9 \\ & & & x_4 & = 1 & \qquad Eqn-11 \end{cases}$$

The augmented and the coefficient matrices of this echelon form are given by:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -.5 & -.5 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The above discussions and examples demonstrate that the three basic operations that we used to reduce a system of linear equations to a row-echelon form, can be translated to a version for matrices.

Definition 1.2.3. By an **elementary row operation** on a matrix we mean one of the following three:

1. Interchange two rows.

2. Multiply a row by a nonzero constant.

3. Add a multiple of a row to another row.

Two matrices are said to be **row-equivalent** if one can be obtained from another by application of a sequence of elementary row operations. Two row-equivalent matrices, correspond to two equivalent system of equations.

Now we define the matrix version of row-echelon form:

Definition 1.2.4. A matrix is said to be in **row-echelon form**, if it has the following properties:

1. All rows consiting entierly of zeros occur at the bottom of the matrix.
2. For each non-zero row, first nonzero entry is 1 (called the **leading 1**).
3. For each successive nonzero rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in row-echelon form is said to be **in reduced row-echelon form**, if every column that has a leading 1 has zeros in evey position above and below the leading 1.

Theorem 1.2.5. *Suppose M is a matrix. Then, M is row-equivalent to a matrix B, which is in row-echelon form.* We gave (see below definition 1.2.2 above) the augmented matrix of the system in exercise 1.1.9 and that of the equivalent system in row-echelon form.

Read [Textbook, Example 4, p 16] for examples of matrices in row echelon form.

Definition 1.2.6. Consider a system of linear equations, as in definition 1.2.2. The method of solving this system by **Gaussian elimination with back-substitution** equation is described as follows:

1. Write the augmented matrix of the system.
2. Use the elemetary row operations tto reduce the augmented matrix to a matrix in row-echelon form.
3. Write the linear system corresponding to the row-echeclon matrix and solve by back-substitution.

Exercise 1.2.7 (Ex 1.1.9, use GE)**.** We will use the method of Gaussian elimination with beck-substitution to solve exercise 1.1.9, using analogous steps. Recall the system:

$$\left\{\begin{array}{lllll} x_1 & & & +3x_4 & = 4 \\ & 2x_2 & -x_3 & -x_4 & = 0 \\ & 3x_2 & & -2x_4 & = 1 \\ 2x_1 & -x_2 & +4x_3 & & = 5 \end{array}\right. \qquad \begin{array}{l} Eqn-1 \\ Eqn-2 \\ Eqn-3 \\ Eqn-4 \end{array}$$

The augmented matrix is:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 2 & -1 & 4 & 0 & 5 \end{pmatrix}$$

Subtract 2 times row-1 from row-4:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 0 & -1 & 4 & -6 & -3 \end{pmatrix}$$

Multiply row-2 by .5:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 0 & -1 & 4 & -6 & -3 \end{pmatrix}$$

Subtract 3 times row-2 from row-3 and add row-2 to Eqn-4:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 0 & 1.5 & -.5 & 1 \\ 0 & 0 & 3.5 & -6.5 & -3 \end{pmatrix}$$

Myltiply row-3 by $\frac{2}{3}$:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 3.5 & -6.5 & -3 \end{pmatrix}$$

Subtract 3.5 times row-3 from row-4:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & -\frac{16}{3} & -\frac{16}{3} \end{pmatrix}$$

Multiply row-4 by $-\frac{3}{16}$:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The above is a matrix in row-echelon form row-equivalent to the augmented matrix. Now the system of linear equations corresponding this row-echelon matrix is

$$\begin{cases} x_1 & & & +3x_4 & = 4 \\ & x_2 & -.5x_3 & -.5x_4 & = 0 \\ & & x_3 & -\frac{1}{3}x_4 & = \frac{2}{3} \\ & & & x_4 & = 1 \end{cases}$$

By back-substitution:

$$x_4 = 1, \quad x_3 = \frac{2}{3} + \frac{1}{3} = 1, \quad x_2 = 1, \quad x_1 = 1.$$

Read [Textbook, Example 5, 6, p 19-] for other such use of this method.

Definition 1.2.8. A matrix in row-echelon form is said to be in **Gauss-Jordan form**, if all the entries above leading entries are zero.

The method of Gaussian elimination with back substitution to solve system of linear equations can be refined by first further reducing the augmented matrix to a Gauss-Jordan form and work with the sytem corresponding to it. This method is called **Gauss-Jordan elimination** method of solving linear sytems.

Consider exercise 1.2.7, the matrix in the row-echelon form, equivalent to the augmented matrix, is

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -.5 & -.5 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

All the entries above the leading 1 in row 2 is zero. So, we try to achieve the same above the leading 1 in row 3. Add .5 times row 3 to row 2:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Now we want to get zeros above the leading 1 in row 4. Subtract 3 times the row 4 from row 1; add $\frac{2}{3}$ times the row 4 from row 2; add $\frac{1}{3}$ times the row 4 from row 3:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

This matrix is in Gauss-Jordan form. The system of linear equation corresponding to this one is:

$$\begin{cases} x_1 & & & & = 1 \\ & x_2 & & & = 1 \\ & & x_3 & & = 1 \\ & & & x_4 & = 1 \end{cases}$$

So, the solution to the system is:

$$x_4 = 1, \qquad x_3 = 1, \qquad x_2 = 1, \qquad x_1 = 1.$$

Read [Textbook, Example 7, 8 p 22-23] for more on Gauss-Jordan elimination.

Remark. If you feel comfortable working with matrices, it is best to reduce a system to Gauss-Jordan, instead of only to row-echelon form.

Exercise 1.2.9 (Ex. 30, p 26)**.** Solve the following using Gaussian elemination or Gauss-Jordan elemination:

$$\begin{cases} 2x_1 & -x_2 & +3x_3 & = 24 & \qquad Eqn-1 \\ & 2x_2 & -x_3 & = 14 & \qquad Eqn-2 \\ 7x_1 & -5x_2 & & = 6 & \qquad Eqn-3 \end{cases}$$

The augmented matrix is

$$\begin{pmatrix} 2 & -1 & 3 & 24 \\ 0 & 2 & -1 & 14 \\ 7 & -5 & 0 & 6 \end{pmatrix}$$

Divide first row by 2 and divide second row by 2:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 7 & -5 & 0 & 6 \end{pmatrix}$$

Subtract 7 times first row from third row:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & -\frac{3}{2} & -\frac{21}{2} & -78 \end{pmatrix}$$

Add $\frac{3}{2}$ times second row to the third row:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & -\frac{45}{4} & -\frac{135}{2} \end{pmatrix}$$

Multiply third row by $-\frac{4}{45}$:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

The above matrix is in row-echelon form. So, we can use back substitution and solve the system. The system corresponding to this matrix is:

$$\begin{cases} x_1 & -\frac{1}{2}x_2 & +\frac{3}{2}x_3 & = 12 \\ & x_2 & -\frac{1}{2}x_3 & = 7 \\ & & x_3 & = 6 \end{cases}$$

By back-substitution:

$$x_3 = 6, \quad x_2 = 7 + \frac{1}{2}6 = 10, \quad x_1 = 12 - \frac{3}{2}6 + \frac{1}{2}10 = 8.$$

Alternately, *we could reduce the row-echelon matrix*

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

to a Gauss-Jordan form. We will do this. To do this add $\frac{1}{2}$ time the second row to the first:

$$\begin{pmatrix} 1 & 0 & 1.25 & 15.5 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

Subtract 1.25 times third rwo from the first:

$$\begin{pmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

Now add .5 time the third row to the second:

$$\begin{pmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

This matrix is in Gauss-Jordan form. The system of liner equations corresponding to this matrix is:

$$\begin{cases} x_1 & & & = 8 \\ & x_2 & & = 10 \\ & & x_3 & = 6 \end{cases}$$

This gives the solution of our system.

Exercise 1.2.10 (Ex. 32, p 26)**.** Solve the following using Gaussian elemination or Gauss-Jordan elemination:

$$\begin{cases} 2x_1 & & +3x_3 & = 3 & \qquad Eqn-1 \\ 4x_1 & -3x_2 & +7x_3 & = 5 & \qquad Eqn-2 \\ 8x_1 & -9x_2 & 15x_3 & = 10 & \qquad Eqn-3 \end{cases}$$

The augmented matrix is

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{pmatrix}$$

We will reduce this matrix to row-echelon form. Subtract 2 times first row from second row and subtract 4times first row from 3rd row:

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & -9 & 3 & -2 \end{pmatrix}$$

Subtract 3 times the second row from third:

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Divide first row by 2 and second row by -3:

$$\begin{pmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix is in rwo-echelon form. The sytem corresponding to thsi equation is:

$$\begin{cases} x_1 & & +\frac{3}{2}x_3 & = \frac{3}{2} \\ & x_2 & -\frac{1}{3}x_3 & = \frac{1}{3} \\ & & & 0 = 1 \end{cases}$$

The last equation is absurd. So, the sytem is inconsistent.

Exercise 1.2.11 (Ex. 34, p 26)**.** Solve the following using Gaussian elemination or Gauss-Jordan elemination:

$$\begin{cases} x & +2y & +z & = 8 \\ -3x & -6y & -3z & = -21 \end{cases}$$

The augmented matrix is

$$\begin{pmatrix} 1 & 2 & 1 & 8 \\ -3 & -6 & -3 & -21 \end{pmatrix}$$

Add 3 times first row to the second row:

$$\begin{pmatrix} 1 & 2 & 1 & 8 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

The above matrix is in row-echelon form. The corresponding system of linear equations is

$$\begin{cases} x & +2y & +z & = 8 \\ & & 0 & = 3 \end{cases}$$

The last equation is absurd. So, the system is inconsistent.

Exercise 1.2.12 (Ex. 44, p26)**.** Solve the homogeneous linear system corresponding to the coefficient matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Since it is a homogeneous sytems have all the constants zero and they system is:

$$\begin{cases} x_1 & & & = 0 \\ & x_2 & +x_3 & = 0 \end{cases}$$

The system is already in row-echelon form. So, by back substitution:

$$x_2 = -x_3, \quad x_1 = 0.$$

With $x_3 = t$ a paramentric solution is

$$x_1 = 0, x_2 = -t, x_3 = t.$$

Note that this is a four variable problem and unknowns are x_1, x_2, x_3, x_4. The variable x_4 does not appear in these equations. So for any $x_4 = s$ for any s, for each solutions above. So, final parametric solution is

$$x_1 = 0, x_2 = -t, x_3 = t, x_4 = s$$

where s, t are paramenters.

Exercise 1.2.13 (Ex. 50, p27)**.** Consider the system of linear equations.

$$\begin{cases} x & +y & & = 0 & \qquad Eqn - 1 \\ & y & +z & = 0 & \qquad Eqn - 2 \\ x & & +z & = 0 & \qquad Eqn - 3 \\ ax & +by & +cz & = 0 & \qquad Eqn - 4 \end{cases}$$

Find the values of a, b, c such that the system has (a) a unique solution, (b) no solution (c) an infinite number of solution.

Solution: The augmented matrix of the equation:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ a & b & c & 0 \end{pmatrix}$$

Subtract 1 times first row from third and a times first row from fourth:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & b-a & c & 0 \end{pmatrix}$$

Add second row to third:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & b-a & c & 0 \end{pmatrix}$$

Divide third row by 2:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & b-a & c & 0 \end{pmatrix}$$

Add $(a-b)$ second row to fourth:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c+a-b & 0 \end{pmatrix}$$

Subtract $c+a-b$ times third row from fourth: i

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix is in row-echelon form. The corresponding liner system is:

$$\begin{cases} x & +y & & = 0 \\ & y & +z & = 0 \\ & & z & = 0 \\ & & 0 & = 0 \end{cases}$$

The system is consistent for all values of a, b, c, and by back substitution the sytem has unique solution $x = y = z = 0$.